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Inequalities for Derivatives of Polynomials of Special Type*

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In this work some inequalities of G. G. Lorentz [1] are rendered in sharp form and prove to generalize a result of P. Erdös [2]. The improved version seems to be a new formulation for inequalities of this type.

Before stating the inequalities, let us introduce some terminology: P_n will denote the class of polynomials of degree less than or equal to n, while Π_n will denote the set of polynomials of the form $p = \sum_0^n a_k q_{nk}$ with $a_k \geqslant 0$ $k = 0, 1,..., n$, where $q_{nk}(x) = x^k(1 - x)^{n-k}$. Elements of Π_n are called *polynomials with positive coefficients* (in x and $1 - x$) by Lorentz. All inequalities will be stated for the interval [0, 1]. We put

$$\| p \| = \max\{| p(x)| : 0 \leqslant x \leqslant 1\}.$$

The following theorem is due to Lorentz [1].

THEOREM 1. *For each $r = 1, 2,...$ there exists a constant C_r for which*

$$\| p^{(r)} \| \leqslant C_r n^r \| p \| \qquad \text{if} \quad p \in \Pi_n , \quad n = 0, 1,....$$

Erdös [2] demonstrated

THEOREM 2. *If $p \in P_n$, $n = 0, 1,...,$ and all zeros of p are real but lie outside* (0, 1), *then*

$$\| p' \| \leqslant en \| p \| .$$

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This is exact in the sense that $\| q'_{n1} \| / n \| q_{n1} \| \to e$. By way of comparison, the inequalities of the brothers Markov for [0, 1] are (see, e.g., [3])

$$\| p' \| \leqslant 2n^2 \| p \| \qquad \text{and} \qquad \| p'' \| \leqslant \tfrac{4}{3} n^2(n^2 - 1) \| p \|$$

if $p \in P_n$, $n = 0, 1, \ldots$.
Thus much is gained by imposing the restrictions of Theorem 1 or 2. In this paper we will prove

THEOREM 3. *Let $t_1(x) = \frac{1}{2}$ and $t_n(x) = x + (1 - 2x)/n$ for $n \geqslant 2$. Then, for $p \in \Pi_n$, $n = 1, 2, \ldots$, and $0 \leqslant x \leqslant \frac{1}{2}$ one has*

$$-2p(x) \leqslant p'(x) \leqslant enp(t_n(x)) \tag{1}$$

$$-2en(n-1)\, p(t_n(x)) \leqslant p''(x) \leqslant 2en(n-1)\, p(t_n(x)). \tag{2}$$

The novetly of this theorem lies in the fact $t_n(x)$ is independent of p. Note that $x \leqslant t_n(x) \leqslant \frac{1}{2}$ if $x \in [0, \frac{1}{2}]$. A theorem of this type (with nonconstant t_n) is impossible for P_n. Similar theorems for Π_n and higher derivatives or for other classes of polynomials remain open for investigation.

We state as corollaries the improved versions of Theorems 1 and 2.

COROLLARY 1. *If $p \in \Pi_n$ and $n \geqslant 1$, then*

$$\| p' \| \leqslant en \| p \| \qquad \textit{and} \qquad \| p'' \| \leqslant 2en(n-1) \| p \|.$$

Further, $\| q'_{n1} \| / n \| q_{n1} \|$, $\| q''_{n1} \| / 2n(n-1) \| q_{n1} \| \to e \quad (n \to \infty)$.

Proof. If $p \in \Pi_n$, then so does $p(1 - x)$, and the first two statements follow from Theorem 3. The last statement is a routine calculation.

COROLLARY 2. *If $p \in P_n (n \geqslant 1)$ is a real polynomial whose zeros lie outside the open disk with center and radius $\frac{1}{2}$, then*

$$\| p' \| \leqslant en \| p \| \qquad \textit{and} \qquad \| p'' \| \leqslant 2en(b-1) \| p \|.$$

Proof. This follows from Corollary 1 and the following observation of Lorentz: For such p one has either p or $-p \in \Pi_n$. To see this we note that $p \in \Pi_n$ and $q \in \Pi_m$ imply $pq \in \Pi_{n+m}$. Now we factor p and use the identities: $x - r = (1 - r)\, x - r(1 - x)$ and

$$(x-r)(x-\bar{r}) = |\, r \,|^2 (1-x)^2 + 2(|\, r - \tfrac{1}{2} \,|^2 - \tfrac{1}{4})\, x(1-x) + |\, 1 - r \,|^2 x^2.$$

The Proof of (1). It is sufficient to prove (1) with p replaced by the q_{nk} because (1) may be recovered by multiplying by $a_k \geqslant 0$ and adding. We have $q'_{nk}(x) = (k - nx)\, x^{k-1}(1 - x)^{n-k-1}$ $k = 0, 1,\ldots n$, $n \geqslant 1$. (1) is trivial for $n = 1, 2$ and we omit the proof. To prove the right inequality, we consider

$$r_{nk} = \frac{q'_{nk}(x)}{nq_{nk}(t)} = \frac{k/n - x}{t(1-t)}\left(\frac{x}{t}\right)^{k-1}\left(\frac{1-x}{1-t}\right)^{n-k-1}, \qquad k = 0, 1,\ldots, n,$$

where $t = x + a/n$, $a = 1 - 2x$. We must prove $r_{nk} \leqslant e$ when $0 \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant k \leqslant n$, $n \geqslant 3$. The proof hinges on the inequality $1 + u \leqslant e^u$, u real, which yields

$$x/t \leqslant \exp(-a/nt) \qquad \text{and} \quad (1-x)/(1-t) \leqslant \exp(a/n(1-t)).$$

Since $r_{n0} \leqslant 0$, we pass to $k \geqslant 1$, to find

$$r_{nn} = 1/t(x/t)^{n-1} \leqslant 1/t \exp(-[(n-1)/n][a/t]) = f_n(x),$$

while for $1 \leqslant k \leqslant n-1$ and $0 \leqslant x \leqslant k/n$, $r_{nk} \leqslant w \exp(-aw + T + S) = U_k$ where $w = (k/n - x)/t(1 - t)$, $T = a(a + 1)/nt$, $S = a(a - 1)/n(1 - t)$. Now $f_n(x)$ increases with x, and attains the value 2 at $\frac{1}{2}$. Hence $r_{nn} \leqslant 2$. Fixing x and regarding k as a continuous variable, we find that U_k increases until $k = k(x) = nx + nt(1 - t)/a$ and decreases thereafter. But $k(x)$ increases with x and $k(1/3) \geqslant n$, so that if $1/3 \leqslant x \leqslant \frac{1}{2}$, $U_k \leqslant U_n \leqslant f_n(x) \leqslant 2$. When $0 \leqslant x \leqslant 1/3$, $U_k \leqslant U_{k(x)} = a^{-1} \exp(-1 + T + S)$. An easy calculation shows $U_{k(x)}$ is logarithmically convex in x, whence $U_{k(x)} \leqslant \max(U_{k(0)}, U_{k(1/3)})$. But $U_{k(0)} = e \geqslant U_{k(1/3)}$, completing the proof of the right inequality. For the left inequality, $q'_{nk}(x)/nq_{nk}(x) = k/(n - x)/x(1 - x) \geqslant -1/(1 - x) \geqslant -2$ if $0 \leqslant x \leqslant \frac{1}{2}$, suffices.

The Proof of (2). As before, it is sufficient to prove (2) with p replaced by q_{nk}. For $0 \leqslant k \leqslant n$, $n \geqslant 2$, $q''_{nk}(x) = n(n-1)\, T_{nk}(x)\, x^{k-2}(1 - x)^{n-k-2}$, where

$$T_{nk}(x) = x^2 - \frac{2k}{n}x + \frac{k(k-1)}{n(n-1)} = \left(x - \frac{k}{n}\right)^2 - \frac{k(n-k)}{n^2(n-1)}.$$

We shall prove

$$-2e \leqslant r_{nk} \leqslant 2e \text{ if } 0 \leqslant x \leqslant \tfrac{1}{2},\ 0 \leqslant k \leqslant n,\ n \geqslant 2, \tag{4}$$

where

$$r_{nk} = \frac{q''_{nk}(x)}{n(n-1)\, q_{nk}(t)} = \frac{T_{nk}(x)}{t^2(1-t)^2}\left(\frac{x}{t}\right)^{k-2}\left(\frac{1-x}{1-t}\right)^{n-k-2},$$

$t = x + a/n$, $a = 1 - 2x$. First we consider some extreme cases. If $n = 2$,

$t = \frac{1}{2}$ identically and (4) is trivial. When $n = 3$, a direct calculation shows $| r_{nk} | \leqslant 9/2$ for $k = 0, 1, 2, 3$ and $0 \leqslant x \leqslant \frac{1}{2}$. By (3),

$$0 \leqslant r_{nn} \leqslant t^{-2} \exp(2 - 1/t) = g(t)$$

and

$$0 \leqslant r_{n0} \leqslant (1 - t)^{-2} \exp(2 - 1/(1 - t)) = g(1 - t).$$

But $g(t) \leqslant g(\frac{1}{2}) = 4$, disposing of $k = 0, n$. Some consideration shows r_{n1} increases with x in $[0, \frac{1}{2}]$ so that

$$-2e < -2(n/(n - 1))^{n-1} \leqslant r_{n1} \leqslant 8((n - 2)/2n) < 4.$$

Writing $r_{n,n-1} = (1 - x - 2/n)/(1 - t)\,(x^{n-3}/t^{n-1})$ we observe that the first quotient decreases in $[0, \frac{1}{2}]$ while the second maximizes at $x=(n-3)/2(n-2)$, so $0 \leqslant r_{n,n-1} \leqslant 4[n/(n-1)]^{n-1}\,[(n-3)/(n-2)]^{n-2}\,[(n-2)^2/(n-1)(n-3)] \leqslant 4$ $(n \geqslant 4)$.

Now we shall concern ourselves with the right side of (4). First we confine our attentions to $0 \leqslant x \leqslant b_n = 1/2(n - 2)$. In this interval r_{n2} decreases while for $3 \leqslant k \leqslant n - 1$ the factor x^{k-2}/t^k of r_{nk} increases and the other factor decreaxes. Thus

$$r_{n2} \leqslant 2[n/(n - 1)]^{n-1} \leqslant 2e,$$

and

$$r_{nk} \leqslant [k(k - 1)/n(n - 1)][n/(n - 1)]^{n-k} (b_n)^{k-2} (2n/3)^k = A_k .$$

Since $A_{k+1} \leqslant A_k$ for $k \geqslant 3$,

$$A_k \leqslant A_3 = (8/9)[(n - 1)/(n - 2)][n/(n - 1)]^{n-1} \leqslant \tfrac{3}{2}e.$$

Now our interest turns to the interval $b_n \leqslant x \leqslant \frac{1}{2}$. Simplifying T_{nk} and using (3) we find $r_{nk} \leqslant w^2 \exp(-aw + T + S) = U_k$, where

$$w = (k - nx)/t(1 - t), \qquad T = a(a + 2)/nt, \qquad S = a(a - 2)/n(1 - t).$$

With fixed x, U_k decreases in $0 \leqslant k \leqslant nx$, increases in $nx \leqslant k \leqslant k(x)$, and decreases for $k \geqslant k(x)$, where $k(x) = nx + 2nt(1 - t)/a$. Since $k(x)$ increases with x and $k(\frac{1}{4}) > n$, we have $U_k \leqslant \max(U_0 , U_n)$ if $\frac{1}{4} \leqslant x \leqslant \frac{1}{2}$, and $U_k \leqslant \max(U_0 , U_{k(x)} , U_n)$ if $b_n \leqslant x \leqslant \frac{1}{4}$. By (3), $U_0 \leqslant g(1 - t) \leqslant 4$ and $U_n \leqslant g(t) \leqslant 4$. Since $b_4 = \frac{1}{4}$ we may take $n \geqslant 5$ when bounding $U_{k(x)}$ in $b_n \leqslant x \leqslant \frac{1}{4}$. But $U_{k(x)}$ is logarithmically convex, which implies

$$U_{k(x)} \leqslant \max(U_{k(b_n)} , U_{k(1/4)}).$$

The latter quantity does not exceed 4, we have $r_{nk} \leqslant 4$ if $b_n \leqslant x \leqslant \frac{1}{2}$, $2 \leqslant k \leqslant n-2$, The proof of the right part of (4) is complete.

Let us turn now to the left inequality in (4). Only $2 \leqslant k \leqslant n-2$ remain. For $k = 2$, $-r_{n2} \leqslant 2(n-2)(1-x)^{n-4}/n^2(n-1)\, t^2(1-t)^{n-2}$. The function of x on the right decreases, so $-r_{n2} \leqslant 2[(n-2)/n][n/(n-1)]^{n-1} \leqslant 2e$. For $3 \leqslant k \leqslant n-2$ we use

$$-r_{nk} \leqslant R_{nk} = \frac{n-3}{n-1}\frac{k}{n^2}\frac{1}{t^2(1-t)^2}\left(\frac{x}{t}\right)^{k-2}\left(\frac{1-x}{1-t}\right)^{n-k-2}.$$

In $0 \leqslant x \leqslant b_n$, R_{nk} increases since each quotient does, whence $R_{nk} \leqslant R_{nk}(b_n) = B_k$. For each $n \geqslant 5$, B_k decreases in $k \geqslant 3$, so $B_k \leqslant B_3$. But $B_3 \leqslant (4/9)[n/(n-2)]^{n-2} < 4$. When $b_n \leqslant x \leqslant \frac{1}{2}$ we apply (3) to obtain

$$R_{nk} \leqslant V_k = \frac{n-3}{n(n-1)\, t(1-t)}\, z \exp(-az + T + S),$$

where $z = k/nt(1-t)$, $T = 2a/nt$, $S = (n-2)\, a/n(1-t)$. As a function of k, V_k increases until $k = k(x) = nt(1-t)/a$ and decreases thereafter. $k(x)$ increases with x and $k(2/5) \geqslant n$, so for $\frac{2}{5} \leqslant x \leqslant \frac{1}{2}$ we have $V_k \leqslant V_n$. But V_n increases with x, taking a maximum at $\frac{1}{2}$ which is less than 4. When $b_n \leqslant x \leqslant \frac{2}{5}$, $V_k \leqslant V_{k(x)}$ and an application of (3) yields

$$V_{k(x)} \leqslant [(n-3)/(n-1)](\mathrm{nat})^{-1} \exp(2a/nt) = h(x).$$

$h(x)$ is logarithmically convex in $0 \leqslant x \leqslant \frac{1}{2}$, so that $h(x) \leqslant \max(h(b_n), h(2/5)) \leqslant 4$ if $n \geqslant 5$ and $b_n \leqslant x \leqslant \frac{2}{5}$. All is done.

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